SMOOTHING LOCALLY FLAT IMBEDDINGS[1]

BY R. C. KIRBY

Communicated by E. Dyer, August 23, 1965

The fundamental imbedding problem for manifolds is to classify the imbeddings of an n-manifold into a q-manifold under ambient isotopy. We announce here that the differentiable and topological cases of this problem for differentiable manifolds are the same if $2q > 3(n+1)$ and $q \geqq 8$.

This follows from Theorem 2 below which states that a locally flat imbedding of a compact differentiable manifold M^n into a differentiable manifold Q^q is ambient isotopic to a differentiable imbedding if $2q > 3(n+1)$ and $q \geqq 8$. Since this ambient isotopy may be chosen arbitrarily close to the identity map, the set of differentiable imbeddings is dense in the set of locally flat imbeddings of M^n in Q^q.

It will then follow that two locally flat imbeddings of M^n into Q^q are ambient isotopic if they are homotopic; hence the classification problem reduces to a problem in homotopy theory.

THEOREM 1. *Let* $f: B^n \to \operatorname{int} Q^q$ *be a locally flat imbedding of the unit* n*-ball into* Q^q. *Such an* f *always extends to* $f: R^q \to \operatorname{int} Q^q$. *Let* C^{n-1} *be a compact differentiable submanifold of* $\partial B^n = S^{n-1}$, *and suppose that* f *is differentiable on a neighborhood of* C^{n-1} *in* B^n. *Let* $q \geqq 7$, $2q > 3(n+1)$ *and* $\epsilon > 0$. *Then there exists an ambient* ϵ*-isotopy* $F_t: Q^q \to Q^q$, $t \in [0, 1]$, *satisfying*

(1) $F_0 = \textit{identity}$,

(2) $F_1 f$ *is differentiable on* $\operatorname{int} B^n$ *and on a neighborhood of* C^{n-1} *in* B^n,

(3) $F_t = \textit{identity}$ *on* $Q - N_\epsilon(f(B^n))$ *and on* $f(R^n - \operatorname{int} B^n)$ *for all* $t \in [0, 1]$,

(4) $|F_t(x) - x| < \epsilon$ *for all* $x \in Q^q$ *and* $t \in [0, 1]$. ($N_\epsilon(X)$) *is the set of points within* ϵ *of* X.)

THEOREM 2. *Let* $f: M^n \to Q^q$ *be a locally flat imbedding such that either* $f(M^n) \subset \operatorname{int} Q^q$ *and* $q \geqq 7$ *or* $f^{-1}(\partial Q^q) = \partial M^n$ *and* $q \geqq 8$. *Let* $2q > 3(n+1)$ *and* $\epsilon > 0$. *Then there exists an ambient* ϵ*-isotopy* $F_t: Q^q \to Q^q$, $t \in [0, 1]$, *satisfying*

(1) $F_0 = \textit{identity}$,

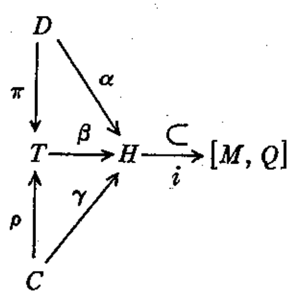
(2) $F_1 f$ *is a differentiable imbedding*,

[1] This is an announcement of a portion of the author's dissertation at the University of Chicago written under Professor Eldon Dyer.

(3) *F_t = identity on $Q - N_\epsilon(f(M^n))$ for all $t \in [0, 1]$,*
(4) *$|F_t(x) - x| < \epsilon$ for all $x \in Q^q$ and $t \in [0, 1]$.*

The proof follows from Theorem 1 by considering the handlebody decomposition of M^n, and smoothing the imbedding of one handle at a time.

Only imbeddings of M^n into Q^q satisfying $f(M^n) \subset \operatorname{int} Q^q$ or $f^{-1}(\partial Q^q) = \partial M^n$ will be considered. Let T be the set of equivalence classes of locally flat imbeddings of M^n into Q^q under equivalence by ambient isotopy. Similarly, let $D(C)$ be the set of equivalence classes of differentiable (combinatorial) imbeddings of M^n into Q^q under equivalence by ambient diffeotopy (ambient combinatorial isotopy). Let H be the homotopy classes of locally flat imbeddings of M^n into Q^q. H is a subset of $[M^n, Q^q]$, the homotopy classes of maps of M^n into Q^q. Then we have the following commutative diagram where the maps are the natural projections.



β is clearly onto for all n and q. Gluck has shown [1] that ρ and γ, and hence β and βi are isomorphisms for $q \geqq 2n+2$. Haefliger has shown [2] that π is a monomorphism and that α is an isomorphism if $2q > 3(n+1)$.

It follows from Theorem 2 that π is also epimorphic if $2q > 3(n+1)$ and either $q \geqq 7$ when $f(M^n) \subset \operatorname{int} Q^q$ or $q \geqq 8$ when $f^{-1}(\partial Q^q) = \partial M^n$. Then π and β are isomorphisms in this range of dimensions.

References

1. H. Gluck, *Embeddings in the trivial range*, Ann. of Math. **81** (1965), 195–210.
2. A. Haefliger, *Plongements différentiables de variétés dans variétés*, Comment. Math. Helv. **36** (1961), 47–82.

University of California, Los Angeles