26

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A calculation of Pin^+ bordism groups

R. C. KIRBY[1] and L. R. TAYLOR[1]

We begin by recalling the definition of the *Pin* and *Spin*-bordism groups. For each integer $n \geq 1$ there are compact Lie groups, $Spin(n)$, $Pin^-(n)$ and $Pin^+(n)$. Atiyah, Bott and Shapiro [ABS], described the groups $Spin(n)$ and $Pin^-(n)$ in terms of the Clifford algebra associated to the negative definite form on $\mathbf{R}^n$. Lam [*L*], describes these as well as $Pin^+(n)$, the group coming from the Clifford algebra associated to the positive definite form on $\mathbf{R}^n$. Another definition is the following. The group $Spin(n)$ is the double cover of the group $SO(n)$. It is a $Z/2$ central extension of $SO(n)$ and is classified by $w_2 \in H^2(BSO(n); Z/2)$: indeed it is the unique non-trivial $Z/2$ central extension. The two groups $Pin^\pm$ are double covers of $O(n)$. They are also $Z/2$ central extensions: Pin^- is classified by $w_2 + w_1^2 \in H^2(BO(n); Z/2)$ and Pin^+ is classified by w_2.

There is a bordism theory of manifolds with *Spin*, Pin^-, or Pin^+ structure, and we use the term bordism groups for the bordism groups of a point. Anderson, Brown and Peterson calculated the *Spin*-bordism groups, [ABP1], and the Pin^--bordism groups, [ABP2]. We complete the story by calculating the Pin^+-bordism groups.

Both the $Pin^\pm$-bordism groups are 2-torsion, and they have cyclic summands of order equal to an arbitrarily high power of 2. Both bordism groups are modules over the *Spin* bordism ring. Of the real projective spaces, the RP^{4k}'s have Pin^+ structures and the RP^{4k+2}'s have Pin^- structures. The other result in this paper is that $Pin^\pm$-bordism, modulo the *Spin* bordism submodule generated by the real projective spaces, is a $Z/2$ vector space.

To describe our results in more detail, recall the 2-local decomposition of the spectrum **MSpin** from [ABP1].

$$\mathbf{MSpin} \to \bigvee_{k \geq 0} \pi(2k)\mathbf{bo}\langle 8k \rangle \bigvee_{k > 0} \pi(2k+1)\mathbf{bo}\langle 8k+2 \rangle \bigvee_{k > 0} \alpha(k)\mathbf{K}(Z/2, k)$$

where $\mathbf{bo}\langle r \rangle$ denotes the spectrum obtained from the usual *BO* spectrum by killing all the homotopy groups in dimensions less than r, and $\mathbf{K}(A, r)$ denotes the Eilenberg–MacLane spectrum with one non-zero homotopy group isomorphic to A

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$\pi_{8n+i} =$	$8n$	$8n+1$	$8n+2$	$8n+3$	$8n+4$	$8n+5$	$8n+6$	$8n+7$
$\mathbf{M}(1) \wedge \mathbf{bo}\langle 2\rangle$	$Z/2 \oplus Z/2$	$Z/2$	$Z/2^{4n+1}$	$Z/2$	$Z/2$	0	$Z/2^{4n+2}$	$Z/2$
$\mathbf{M}(3) \wedge \mathbf{bo}\langle 2\rangle$	$Z/2^{4n-1}$	$Z/2$	$Z/2 \oplus Z/2$	$Z/2$	$Z/2^{4n+2}$	$Z/2$	$Z/2$	0

for $0 \leq i < 8$, $n \geq 0$ *and* $8n + i \geq 3$. *In the case* $n = 0$, $i = 0$ *or* 1,

$$\pi_i(\mathbf{M}(1) \wedge \mathbf{bo}\langle 2\rangle) = \pi_i(\mathbf{M}(3) \wedge \mathbf{bo}\langle 2\rangle) = 0.$$

In the case $n = 0$, $i = 2$,

$$\pi_2(\mathbf{M}(1) \wedge \mathbf{bo}\langle 2\rangle) = \pi_2(\mathbf{M}(3) \wedge \mathbf{bo}\langle 2\rangle) = Z/2.$$

COROLLARY 2. *The top line of the first table, with* $n = 0$, *gives the* Pin^- *bordism groups through dimension* 7; *the second line of the first table, with* $n = 0$, *gives the* Pin^+ *bordism groups through dimension* 7.

An alternate calculation of these bordism groups through dimension 4 is given in [KT]. While trying to understand these low-dimensional calculations, we were led to the general results presented here. The proofs will be given in the second section and a short table of the bordism groups is included at the end of the paper.

Notice that Pin^- bordism is a $Z/2$ vector space except in dimensions congruent to 2 mod 4. Moreover, RP^n has a Pin^- structure if n is congruent to 2 mod 4. Likewise, Pin^+ bordism is a $Z/2$ vector space except in dimensions congruent to 0 mod 4 and RP^n has a Pin^+ structure if n is congruent to 0 mod 4.

Recall some facts about the structure of the *Spin* bordism ring. The $\mathbf{bo}\langle\ \rangle$ factors are indexed by partitions. For a fixed $n = 8k$ we have a different $\mathbf{bo}\langle 8k\rangle$ for each partition, J, of $2k$ such that J has no 1's in it. For any partition, let $n(J)$ denote the sum of the elements of J, or in other words, $n(J)$ is the integer for which J is a partition. The $\mathbf{bo}\langle 8k + 2\rangle$'s are indexed by the partitions, J, with no 1's for which $n(J) = 2k + 1$. In the sequel, let $\mathbf{bo}\langle J\rangle$ denote $\mathbf{bo}\langle 4n(J)\rangle$ if $n(J)$ is even or $\mathbf{bo}(4n(J) - 2\rangle$ if $n(J)$ is odd. There is also a copy of $\mathbf{bo}\langle 0\rangle$. There are elements M_J in dimensions $4n(J)$, where J is a partition of $n(J)$ with no 1's. These manifolds satisfy the condition that in our fixed decomposition of **MSpin**, the bordism class of M_J is a generator of $\pi_{4n(J)}\mathbf{bo}\langle J\rangle$ and maps to zero in $\pi_{4n(J)}$ of all the other summands.

Let $X(J, n) = RP^n \times M_J$ if $n(J)$ is even. If n is even, fix a $Pin^{\pm}$ structure on RP^n and consider $X(J, n)$ as an element of $Pin^{\pm}$ bordism. If $n(J)$ is odd, $RP^n \times M_J$ will be divisible by 2 in the corresponding *Pin* bordism group, so let $X(J, n)$ denote an element in $Pin^{\pm}$ bordism such that $2X(J, n) = RP^n \times M_J$. Note that for Pin^+ bordism we are asserting that $M_J = M_J \times RP^0$ is divisible by 2. Let $C(J, 2n)$ denote a cyclic group whose order is the order of the element $X(J, 2n)$ in the appropriate *Pin* bordism group. There are natural maps $C(J, 4n) \to MPin^+_{4n(J)+4n}$ and $C(J, 4n+2) \to MPin^-_{4n(J)+4n+2}$.

THEOREM 3. *The order of* $X(J, 2n)$ *is given as follows*:

	$2n = 8k$	$2n = 8k+2$	$2n = 8k+4$	$2n = 8k+6$
$n(J)$ even	2^{4k+1}	2^{4k+3}	2^{4k+4}	2^{4k+4}
$n(J)$ odd	2^{4k+2}	2^{4k+2}	2^{4k+3}	2^{4k+5}

The sum of the natural maps

$$\bigoplus_{J,n} C(J, 4n) \to MPin^+_*$$

is injective with image a summand: the complementary summand is a $Z/2$ *vector space. The sum of the natural maps*

$$\bigoplus_{J,n} C(J, 4n+2) \to MPin^-_*$$

is injective with image a summand: the complementary summand is a $Z/2$ *vector space. In both sums,* $n \geq 0$ *and* J *runs over all partitions with no* 1*'s.*

COROLLARY 4. *The* Pin^+ *bordism groups, modulo the Spin bordism submodule generated by the* RP^{4n}, *are* $Z/2$ *vector spaces. The* Pin^- *bordism groups, modulo the Spin bordism submodule generated by the* RP^{4n+2}, *are* $Z/2$ *vector spaces.*

Finally, we pause to consider the standard question of the image of Pin^+ bordism in unoriented bordism, denoted $\mathcal{N}_*$. Using the techniques of Anderson, Brown and Peterson [ABP2], we show

COROLLARY 5. *The image of the natural map* $MPin^+_* \to \mathcal{N}_*$ *equals all bordism classes all of whose Stiefel–Whitney numbers involving* $w_2(\tau)$ *vanish, where* τ *denotes the tangent bundle.*

After this paper was submitted, we learned of the paper of Giambalvo [G], which also calculates $MPin^+$ bordism. Giambalvo does the calculation via the Adams' spectral sequence and arrives at the same answer we do. He also attempted to analyse the role of the RP^{2n}'s in Pin^+ and Pin^- bordism, using the map ψ described below, but his results differ considerably from ours. Specifically, we claim that the order of RP^{8n+4} in Pin^+ bordism is 2^{8n+4} and that his Corollary 3.5 is wrong (see the discussion preceding Theorem 3). The table on page 399 is also incorrect: the factor corresponding to $\mathbf{M}(2) \wedge \mathbf{bo}\langle 8\rangle$ is missing and the Z_2^8 should be $Z/2^8$.

We would like to thank S. Stolz for numerous conversations on the subject of *Pin* bordism.

Proofs

We begin with two lemmas to reduce the calculation to a diagram chase.

LEMMA 6. *The* ith Pin^+ *bordism group is isomorphic to*

$$\pi_i(\mathbf{MSpin} \wedge \mathbf{M}(4k+3)) \quad \textit{for any } k \geq 0.$$

The ith Pin^- *bordism group is isomorphic to*

$$\pi_i(\mathbf{MSpin} \wedge \mathbf{M}(4k+1)) \quad \textit{for any } k \geq 0.$$

In both cases, the usual transversality construction gives the isomorphism.

Proof. Let us begin with the Pin^+ case. Standard transversality constructions identify $\pi_i(\mathbf{MSpin} \wedge \mathbf{M}(4k+3))$ with the bordism theory of i-dimensional manifolds with a *Spin* structure on the bundle $\tau \oplus (4k+3)\det(\tau)$, where τ is the tangent bundle to the manifold and $\det(\tau)$ is the determinant line bundle. It is easy to check that for any bundle η, 4η has a canonical *Spin* structure, so the above bordism theory is equivalent to the bordism theory of i-dimensional manifolds with a *Spin* structure on the bundle $\tau \oplus 3\det(\tau)$. Next one can compute that any bundle η has a Pin^+ structure iff $\eta \oplus 3\det(\eta)$ has a *Spin* structure, and, since this is a universal relation, one can set up a one-to-one correspondence between *Spin* structures on $\eta \oplus 3\det(\eta)$ and Pin^+ structures on η. Hence our bordism theory is equivalent to the bordism theory of i-dimensional manifolds with a Pin^+ structure on the tangent bundle.

The Pin^- case is entirely similar. □

Let $\mathbf{M}(Z/2, 0) = e^0 \cup e^1$ with attaching map of degree 2 and denote the homotopy ith group of $\mathbf{MSpin} \wedge \mathbf{M}(Z/2, 0)$ by $(\mathbf{MSpin} \wedge Z/2)_i$. These groups can largely be calculated by applying *Spin* bordism to the cofibration sequence $S^0 \xrightarrow{\times 2} S^0 \to \mathbf{M}(Z/2, 0)$, since the degree 2 map on S^0 induces multiplication by 2 on the *Spin* bordism groups.

These groups have an interpretation as $Z/2$-*Spin* bordism. This is the bordism theory consisting of a manifold M with a codimension-one submanifold N; an orientation on $M - N$ which does not extend across any component of N; an orientation of the normal bundle of N in M; a *Spin* structure on $M - N$; a *Spin* structure on N; and diffeomorphisms which preserve the *Spin* structures from N to the boundary components of $M - N$. We do not need this interpretation in the sequel.

LEMMA 7. *There exists a cofibration sequence*

$$\mathbf{M}(Z/2, 0) \to \mathbf{M}(2r-1) \to \Sigma^2\mathbf{M}(2r+1) \tag{8}$$

Hence we get long exact sequences

$$\cdots \to (\mathbf{MSpin} \wedge Z/2)_i \to MPin_i^+ \xrightarrow{\psi} MPin_{i-2}^- \to \cdots$$

$$\cdots \to (\mathbf{MSpin} \wedge Z/2)_i \to MPin_i^- \xrightarrow{\psi} MPin_{i-2}^+ \to \cdots$$

In both cases, the map ψ is defined by starting with a manifold M, finding a submanifold $N \subset M$ dual to ω_1, and then forming the transverse intersection, $N \cap N$. Notice that ψ can also be described by taking the natural map $\psi: \mathbf{M}(r) \to \Sigma^2\mathbf{M}(r+2)$ and smashing it with **MSpin**. *In particular, the two exact sequences above decompose in the same way that* **MSpin** *does.*

Proof. Recall that $T(r\xi) = RP^\infty/RP^{r-1}$. Indeed, $RP^n \subset RP^{n+r}$ with normal bundle $r\xi|_{RP^n}$. Hence we have a map $RP^{n+r} \to T(r\xi|_{RP^n})$ and the composite $RP^n \subset RP^{n+r} \to T(r\xi|_{RP^n})$ is the zero-section. Hence a copy of RP^{r-1} disjoint from RP^n in RP^{n+r} is null-homotopic in $T(r\xi|_{RP^n})$, so we get a map $RP^{n+r}/RP^{r-1} \to T(r\xi|_{RP^n})$ which is easily checked to be a homotopy equivalence.

The cofibration sequence is now clear since RP^{2r}/RP^{2r-2} is homotopy equivalent to $T((2r-1)\xi|_{RP^1})$ and this is $\Sigma^{2r-1}\mathbf{M}(Z/2, 0)$.

The description of the map ψ also follows. Consider a *Spin* boundary M^{m+2r-1} and a map $f: M \to T((2r-1)\xi)$. The map ψ sends f to the composite $M \to T((2r+3)\xi)$ of f and the map $g: T((2r+1)\xi) \to T((2r+3)\xi)$. To see what happens to the underlying *Pin* manifolds, we can assume that f lands in $T((2r-1)\xi|_{RP^N})$ for some large N, and we get a cofibration sequence like (8) but taking place inside of RP^{N+2r+1} instead of RP^∞. We make the new map transverse to the zero-section to get out *Pin* manifold, P. The map g becomes a map $g: T((2r+1)\xi|_{RP^N}) \to T((2r+3)\xi|_{RP^{N-2}})$, so to get $\psi(P)$ we make the map

The other cases are similar so we only discuss the key points. Begin with the next case, $8k+2$ and start with $k=0$. This means we are trying to identify RP^2 in $MPin_2^- = Z/8$. Applying ψ and consulting the first table from Proposition 9, we see that it is a generator. We can now use induction and the four-fold iterate of ψ to handle the case $n(J)$ even. In the case $n(J)$ is odd, we need to identify $M_J \times RP^2$. It lives in a group of order 4, and table one of Proposition 10, shows that ψ is an isomorphism, so $M_J \times RP^2$ is of order 2 in Pin^- bordism, since M_J has order 2 in Pin^+ bordism. Hence we define $X(s, J, 8k+2)$ as above using the four-fold iterate of ψ. The cases $8k+4$ and $8k+6$ are done in the same way.

Now let us define $X(J, 2n) = Y(J, 2n)$ if $n(J)$ is even; for $n(J)$ odd, define $X(J, 2n) = X(2^{\phi(2n)+1} - (2n+1), J, 2n)$. From the above discussion, we know the orders of each of the $X(J, 2n)$'s: let $C(J, 2n)$ denote a cyclic group of this order with a fixed generator and map $C(J, 2n)$ to $MPin^{\pm}$ by sending the fixed generator to $X(J, 2n)$. We get maps

$$\oplus_{J,n} C(J, 4n) \to MPin_*^+ \text{ and } \oplus_{J,n} C(J,\ 4n+2) \to MPin_*^- .$$

For n fixed we see from above that $\oplus_{J,n} C(J, 4n) \to MPin_*^+$ and $\oplus_{J,n} C(J, 4n+2) \to MPin_*^-$ are split injective. Theorem 3 asserts that these maps are still split injective when we also sum over the n.

We do the Pin^+ case. Fix a dimension $r = 8k$. Note that $C(J, 4n)$ lands in dimension r iff $r = 4n(J) + 4n$. If $n(J)$ is even, then $C(J, 4n)$ has order 2^{2n+1} and if $n(J)$ is odd, $C(J, 4n)$ has order 2^{2n+2}. In particular, two $C(J, 4n)$'s which land in the same dimension and have the same order have the same n and the same $n(J)$. If $r = 8k+4$ we get different numbers but the same conclusion. Finally note that both $\oplus_{r=4n(J)+4n} C(J, 4n)$ and $Mpin_r^+$ have the same number of $Z/2^k$ summands for all $k > 1$, and if we restrict the map $\oplus_{r=4n(J)+4n} C(J, 4n) \to MPin_r^+$ to the summands of order 2^k we get a split injection. It is an elementary algebra exercise to verify that this means that the map is a split injection and the complementary summand is a $Z/2$ vector space.

The Pin^- case is entirely similar.

The proof of Corollary 5

We begin with a general discussion of characteristic numbers. Let BG be a space such as BSO, $BPin^+$, etc. equipped with a map to BO. Let M be a manifold with a G structure; i.e. the tangent bundle map $M \to BO$ has a fixed lift to a map $\tau : M \to BG$. Then M^n determines a homomorphism $H^n(BG; Z/2) \to Z/2$ given by sending $x \in H^n(BG; Z/2)$ to $\tau^*(x)$ evaluated on the fundamental class of M. This defines a homomorphism $T : \Omega_n^G \to \text{Hom}\,(H^n(BG; Z/2), Z/2)$. If we let $M(G)$ denote the Thom spectrum for the inverse to the universal bundle over BO pulled-back to

BG, the Thom isomorphism shows that we can equally regard T as a homomorphism $T:\Omega_n^G \to \mathrm{Hom}\,(H^n(M(G);Z/2),Z/2)$. If a homomorphism $b: H^n(M(G);Z/2)\to Z/2$ is to be in the image of T, then $b(ax)=0$ for any a in the mod 2 Steenrod algebra of dimension at least 1 and any $x\in H^*(M(G);Z/2)$. If we let $\mathscr{A}$ denote the mod 2 Steenrod algebra, we can turn $Z/2$ into an $\mathscr{A}$ module by letting all the Sq^i act trivially. Then $\mathrm{Hom}_{\mathscr{A}}\,(H^n(M(G);Z/2),Z/2)\subset \mathrm{Hom}\,(H^n(M(G);Z/2),Z/2)$ is precisely the set of homomorphisms satisfying our condition and Condition P of [ABP2] merely says that the image of T is precisely $\mathrm{Hom}_{\mathscr{A}}\,(H^n(M(G);Z/2),Z/2)$. (It is also true that $\mathrm{Hom}_{\mathscr{A}}\,(H^n(M(G);Z/2),Z/2)=E_2^{0,n}(M(G))$ in the Adams spectral sequence for $\pi_*(M(G))$. Moreover, $E_\infty^{0,n}(M(G))\subset E_2^{0,n}(M(G))$ is precisely the image of T. Hence the collapse of the Adams spectral sequence is sufficient for $M(G)$ to have Property P.)

Now $\mathrm{Hom}_{\mathscr{A}}\,(H^n(M(G);Z/2),Z/2)$ behaves like any other Hom, so we can apply it to the short exact sequences of cohomology groups coming from (8). It is not hard to see directly that $E_2^{0,r}(\mathbf{M}(Z/2,0)\wedge \mathbf{bo}\langle 0\rangle)=Z/2$ if $r=0$; $E_2^{0,r}(\mathbf{M}(Z/2,0)\wedge \mathbf{bo}\langle 2\rangle)=Z/2$ if $r=2$ and both groups are 0 otherwise. Theorem 4.4 of [ABP2] says that $E_2^{0,r}(\mathbf{M}(1)\wedge \mathbf{bo}\langle 0\rangle)=Z/2$ if $r=0$ or $r\equiv 2 \pmod 4$; $E_2^{0,r}(\mathbf{M}(1)\wedge \mathbf{bo}\langle 2\rangle)=Z/2$ if $r=2$ or $r\equiv 0 \pmod 4$ and both groups are 0 otherwise. One can also check by hand that $E_2^{0,r}(\mathbf{M}(3)\wedge \mathbf{bo}\langle 0\rangle)=Z/2$ if $r=0$ and is 0 for $r<3$ and that $E_2^{0,r}(\mathbf{M}(3)\wedge \mathbf{bo}\langle 2\rangle)=Z/2$ if $r=2$ and is 0 otherwise for $r<5$. By comparing the two exact sequences coming from (8) we can compute $E_2^{0,r}(\mathbf{M}(3)\wedge \mathbf{bo}\langle 0\rangle)$ and $E_2^{0,r}(\mathbf{M}(3)\wedge \mathbf{bo}\langle 2\rangle)$. More importantly, we can see that $\psi: E_2^{0,r}(\mathbf{M}(1)\wedge \mathbf{bo}\langle 0\rangle)\to E_2^{0,r-2}(\mathbf{M}(3)\wedge \mathbf{bo}\langle 0\rangle)$ and $\psi: E_2^{0,r}(\mathbf{M}(1)\wedge \mathbf{bo}\langle 2\rangle)\to E_2^{0,r-2}(\mathbf{M}(3)\wedge \mathbf{bo}\langle 2\rangle)$ are both epic. Since $\mathbf{M}(1)\wedge \mathbf{bo}\langle 0\rangle)$ and $\psi: E_2^{0,r}(\mathbf{M}(1)\wedge \mathbf{bo}\langle 2\rangle)\to E_2^{0,r-2}(\mathbf{M}(3)\wedge \mathbf{bo}\langle 2\rangle)$ are both epic. Since $\mathbf{M}(1)\wedge \mathbf{bo}\langle 0\rangle$ and $\mathbf{M}(1)\wedge \mathbf{bo}\langle 2\rangle$ satisfy Property P by [ABP2], this shows that $\mathbf{M}(3)\wedge \mathbf{bo}\langle 0\rangle$ and $\mathbf{M}(3)\wedge \mathbf{bo}\langle 2\rangle$ also satisfy Property P. The Eilenberg–MacLane summands also satisfy Property P, hence so does $MPin^+$.

Since $H^*(BO;Z/2)\to H^*(BPin^+;Z/2)$ is onto, it follows formally that a manifold, M^n, is unoriented bordant to a Pin^+ manifold iff all the characteristic numbers in the kernel of $H^n(BO;Z/2)\to H^n(BPin^+;Z/2)$ vanish on M. This kernel is the ideal in $H^*(BO;Z/2)$ generated by w_2 and its images under the Steenrod algebra: e.g. w_3 is in the kernel. It is always the case however that, if all the characteristic BO-numbers of a manifold which involve $x\in H^r(BO;Z/2)$ vanish, then all the numbers involving $a(x)$ for any $a\in\mathscr{A}$ also vanish. Hence M is bordant to a Pin^+ manifold iff all tangential characteristic numbers involving w_2 vanish.

We may as well finish by remarking that $\mathbf{MSpin}\wedge Z/2$ satisfies Property P and that a manifold is unoriented bordant to an element in $\mathbf{MSpin}\wedge Z/2$ iff all the numbers involving ω_2 and ω_1^2 vanish.

The tables

Here are the promised Pin^+ bordism groups through dimension 95, arranged in two tables. The second table gives $A(n)$, the number of $Z/2$ summands in $MPin_n^+$. The first table gives numbers $\pi(n)$ which enable us to find the other summands in dimensions congruent to 0 mod 4. For $MPin_{8n+4}^+$, the summands of order greater than 2 are $\oplus\pi(i)Z/2^{4n+4-2i}$ beginning with $i=0$ and continuing until $4n+4-2i=2$. For $MPin_{8n+8}^+$, the summands of order greater than 2 are $\oplus\pi(i)Z/2^{4n+5-2i}$ beginning with $i=0$ and continuing until $4n+4-2i=3$. As an example, $28 = 8 \cdot 3 + 4$ so $MPin_{28}^+ = 4Z/2 \oplus (1Z/2^{16} \oplus 0Z/2^{14} \oplus 1Z/2^{12} \oplus 1Z/2^{10} \oplus 2Z/2^8 \oplus 2Z/2^6 \oplus 4Z/2^4 \oplus 4Z/2^2)$

n	$\pi(n)$	n	$\pi(n)$	n	$\pi(n)$	n	$\pi(n)$	n	$\pi(n)$	n	$\pi(n)$
0	1	4	2	8	7	12	21	16	55	20	137
1	0	5	2	9	8	13	24	17	66	21	165
2	1	6	4	10	12	14	34	18	88	22	210
3	1	7	4	11	14	15	41	19	105	23	253

n	$A(n)$	n	$A(n)$	n	$A(n)$	n	$A(n)$	n	$A(n)$	n	$A(n)$	n	$A(n)$	n	$A(n)$
0	1	12	0	24	6	36	17	48	113	60	394	72	1556	84	4965
1	0	13	1	25	5	37	34	49	130	61	526	73	1764	85	5843
2	1	14	1	26	20	38	41	50	244	62	606	74	2440	86	6541
3	1	15	0	27	17	39	27	51	222	63	548	75	2423	87	6605
4	0	16	2	28	4	40	43	52	152	64	673	76	2224	88	7536
5	0	17	1	29	12	41	49	53	220	65	771	77	2694	89	8412
6	0	18	8	30	15	42	109	54	258	66	1150	78	3041	90	10515
7	0	19	7	31	8	43	96	55	218	67	1114	79	2995	91	10814
8	1	20	1	32	16	44	54	56	281	68	959	80	3475	92	10730
9	0	21	4	33	17	45	89	57	324	69	1209	81	3907	93	12365
10	3	22	5	34	48	46	106	58	534	70	1378	82	5103	94	13750
11	3	23	2	35	41	47	81	59	503	71	1310	83	5168	95	14135

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Department of Mathematics
University of California, Berkely
Berkeley, California, USA

Department of Mathematics
University of Notre Dame
Notre Dame, Indiana, USA

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